PHYS 231 - Sept. 20, 2023

Last Time

If make repeated measurements of a quantity x under the same experimental conditions, then report final result as:

$$\boxed{x = \mu \pm \sigma_\mu}$$

where $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$ is the mean

and $\boxed{\sigma_\mu = \frac{\sigma}{\sqrt{N}}}$ is the standard error

with standard deviation σ given by

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu)^2}$$

Propagation of Errors

If meas. $x_1 \pm \Delta x_1$
$x_2 \pm \Delta x_2$
$\vdots$
$x_N \pm \Delta x_N$

and then calculate $y = f(x_1, x_2, \ldots, x_N)$,

the uncertainty in y is given by:

$$\boxed{\Delta y = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_N}\Delta x_N\right)^2}}$$

Today: Examples...

$$R = \frac{V}{I} \qquad \text{meas.} \quad \begin{matrix} V \pm \Delta V \\ I \pm \Delta I \end{matrix} \Rightarrow \text{Find } \Delta R.$$

First-year

$$\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I}$$

Use new prop. of errors formula:

$$\Delta R = \sqrt{\left(\underbrace{\frac{\partial R}{\partial V}}\Delta V\right)^2 + \left(\underbrace{\frac{\partial R}{\partial I}}\Delta I\right)^2}$$

Need to evaluate:

$$\frac{\partial R}{\partial V} = \frac{\partial}{\partial V}\left(\frac{V}{I}\right) = \frac{1}{I}$$

$$\frac{\partial R}{\partial I} = \frac{\partial}{\partial I}\left(\frac{V}{I}\right) = -\frac{V}{I^2}$$

$$\therefore \Delta R = \sqrt{\left(\frac{\Delta V}{I}\right)^2 + \left(\frac{V \Delta I}{I^2}\right)^2}$$

If $V = 5.25 \pm 0.05\,\text{V}$, $I = 32 \pm 2\,\text{mA}$ $\Big\}$ find $R = 160 \pm 10\,\Omega$

To compare to first-year result let's find

$$\left(\frac{\Delta R}{R}\right)^2 = \frac{1}{R^2}\left(\frac{\Delta V}{I}\right)^2 + \frac{1}{R^2}\left(\frac{V \Delta I}{I^2}\right)^2$$

$$\frac{1}{R^2} = \left(\frac{I}{V}\right)^2$$

$$\left(\frac{\Delta R}{R}\right)^2 = \left(\frac{\cancel{I}}{V}\right)^2\left(\frac{\Delta V}{\cancel{I}}\right)^2 + \left(\frac{\cancel{I}}{\cancel{V}}\right)^2\left(\frac{\cancel{V}\Delta I}{I^{\cancel{2}}}\right)^2$$

$$\boxed{\therefore \frac{\Delta R}{R} = \sqrt{\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2}}$$

Same as first-year rule except w/ a quadrature sum.

Example: $y = A x^n$

A ← constant

First year $\dfrac{\Delta y}{y} = n \dfrac{\Delta x}{x}$

By prop. of errors:

$$\Delta y = \sqrt{\left(\frac{\partial y}{\partial x} \Delta x\right)^2}$$

$$= \left| \frac{\partial y}{\partial x} \Delta x \right|$$

Need to evaluate $\dfrac{\partial y}{\partial x} = \dfrac{\partial}{\partial x}\left(A x^n\right)$

$$= \boxed{n A x^{n-1}}$$

$$\Delta y = \left| n A x^{n-1} \Delta x \right|$$

To compare to first-year rule, find

$$\frac{\Delta y}{y} = \left| \frac{n \cancel{A} x^{n-1} \Delta x}{\cancel{A} x^n} \right| = \left| n \frac{\Delta x}{x} \right|$$

same!

Another example

$$z = x \pm y$$

know $x \pm \Delta x$

$y \pm \Delta y$

Find Δz?

$$\Delta z = \sqrt{\left(\underbrace{\frac{\partial z}{\partial x}}_{1} \Delta x\right)^2 + \left(\underbrace{\frac{\partial z}{\partial y}}_{1} \Delta y\right)^2}$$

Need: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x \pm y) = 1 + 0$

$= 1$

$$\frac{\partial z}{\partial z} = \frac{\partial}{\partial y}(x \pm y) = 0 \pm 1$$

$= \pm 1$

$$\therefore \; \Delta z = \sqrt{(\Delta x)^2 + (\Delta y^2)}$$

Same as first-year rule w/ addition of the quadrature sum.

Another example:

$$y = x \ln x$$

know $x \pm \Delta x$
Find Δy.

No first-year rule for this case.
Need to apply general prop. of errors formula.

$$\Delta y = \sqrt{\left(\frac{\partial y}{\partial x} \Delta x\right)^2} = \left| \frac{\partial y}{\partial x} \Delta x \right|$$

Need $\dfrac{\partial y}{\partial x} = \dfrac{\partial}{\partial x}\left(x \ln x \right)$

$$= \ln x + x\left(\frac{1}{x}\right)$$

$$\therefore \boxed{\Delta y = \left| (\ln x + 1)\,\Delta x \right|}$$

Try to justify/derive the standard error result $\sigma_\mu = \frac{\sigma}{\sqrt{N}}$.

Strategy is to write down expression for the mean & then apply prop. of errors.

Meas. of

$$x_1 \pm \Delta x_1$$
$$x_2 \pm \Delta x_2$$
$$\vdots$$
$$x_N \pm \Delta x_N$$

$$\mu = \frac{1}{N} \sum_{i=1}^{N} x_i = \frac{1}{N}\left(x_1 + x_2 + \ldots + x_N\right)$$

μ depends on N individual measurements, so prop. of errors expression will have N terms under $\sqrt{}$.

uncertainty in μ

$$\sigma_\mu = \sqrt{\left(\underbrace{\frac{\partial \mu}{\partial x_1}}_{\frac{1}{N}} \Delta x_1\right)^2 + \left(\underbrace{\frac{\partial \mu}{\partial x_2}}_{\frac{1}{N}} \Delta x_2\right)^2 + \dots + \left(\underbrace{\frac{\partial \mu}{\partial x_N}}_{\frac{1}{N}} \Delta x_N\right)^2}$$

$$\frac{\partial \mu}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\frac{1}{N}\left(x_1 + x_2 + \dots + x_i + \dots + x_N\right)\right]$$

$$= \frac{1}{N}\left(0 + 0 + \dots + 1 + \dots + 0\right)$$

$$= \frac{1}{N}$$

$$\sigma_\mu = \sqrt{\left(\frac{\Delta x_1}{N}\right)^2 + \left(\frac{\Delta x_2}{N}\right)^2 + \dots + \left(\frac{\Delta x_N}{N}\right)^2}$$

$$= \frac{1}{N}\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + \dots + (\Delta x_N)^2}$$

For an experiment repeated in the same way over & over, each individual trial would have the same uncertainty σ.

$$\sigma_\mu = \frac{1}{N}\sqrt{\sigma^2 + \sigma^2 + \cdots + \sigma^2}$$

$$= \frac{1}{N}\sqrt{N\sigma^2} = \boxed{\frac{\sigma}{\sqrt{N}}}$$

Standard error in the mean.